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# Inner sequences and submodules in the Hardy space over the bidisk

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## Abstract

We deal with infinite sequences of inner functions  $\{q_j\}_{j \geq 0}$  with the property that  $q_j$  is divisible by  $q_{j+1}$ . It is shown that these sequences have close relations to the module structure of the Hardy space over the bidisk. This article is a résumé of recent papers. Some results of this research were obtained in joint work with R. Yang (SUNY).

## 1 Preliminaries

Let  $\mathbb{D}$  be the open unit disk in the complex plane  $\mathbb{C}$ , and let  $H^2(z)$  denote the classical Hardy space over  $\mathbb{D}$  with the variable  $z$ . The Hardy space over the bidisk  $H^2$  is the tensor product Hilbert space  $H^2(z) \otimes H^2(w)$  with variables  $z$  and  $w$ . A closed subspace  $\mathcal{M}$  of  $H^2$  is called a submodule if  $\mathcal{M}$  is invariant under the action of multiplication operators of coordinate functions  $z$  and  $w$ . Let  $R_z$  (resp.  $R_w$ ) denote the restriction of the Toeplitz operator  $T_z$  (resp.  $T_w$ ) to a submodule  $\mathcal{M}$ . The quotient module  $\mathcal{N} = H^2/\mathcal{M}$  is the orthogonal complement of a submodule  $\mathcal{M}$  in  $H^2$ , and let  $S_z$  (resp.  $S_w$ ) denote the compression of  $T_z$  (resp.  $T_w$ ) to  $\mathcal{N}$ , that is, we set  $S_z = P_{\mathcal{N}}T_z|_{\mathcal{N}}$  (resp.  $S_w = P_{\mathcal{N}}T_w|_{\mathcal{N}}$ ) where  $P_{\mathcal{N}}$  denotes the orthogonal projection from  $H^2$  onto  $\mathcal{N}$ .

## 2 Rudin's submodule

Let  $\mathcal{M}$  be the submodule consisting of all functions in  $H^2$  which have a zero of order greater than or equal to  $n$  at  $(\alpha_n, 0) = (1 - n^{-3}, 0)$  for any positive

integer  $n$ . This module was given by Rudin in [1], and he proved that this is not finitely generated. Rudin's submodule can be decomposed as follows (cf. [3]):

$$\mathcal{M} = \sum_{j=0}^{\infty} \oplus q_j(z) H^2(z) w^j,$$

where we set  $b_n(z) = (\alpha_n - z)/(1 - \alpha_n z)$ ,  $q_0(z) = \prod_{n=1}^{\infty} b_n^n(z)$  and  $q_j(z) = q_{j-1}(z)/\prod_{n=j}^{\infty} b_n(z)$  for any positive integer  $j$ .

Regarding this submodule, the following are known (cf. [4]):

$$\sigma_p(S_z) = \{\alpha_n : n \geq 1\}, \quad \sigma_c(S_z) = \{1\}, \quad \sigma_r(S_z) = \emptyset$$

and

$$\| [R_z^*, R_w] \|_2^2 = \sum_{j=1}^{\infty} \left( 1 - \prod_{n=j}^{\infty} (1 - n^{-3})^2 \right).$$

Moreover, we have obtained the following in [2]:

$$\sigma_p(S_w) = \{0\}, \quad \sigma_c(S_w) = \overline{\mathbb{D}} \setminus \{0\}, \quad \sigma_r(S_w) = \emptyset$$

and

$$\begin{aligned} \| [S_z^*, S_w] \|_2^2 &= \sum_{j=1}^{\infty} \left( 1 - \prod_{n=j}^{\infty} (1 - n^{-3})^{2(n-j)} \right) \left( 1 - \prod_{n=j}^{\infty} (1 - n^{-3})^2 \right) \\ &= -1 + \sum_{j=1}^{\infty} \left( 1 - \prod_{n=j}^{\infty} (1 - n^{-3})^2 \right). \end{aligned}$$

### 3 Inner sequences

**Definition 1** An infinite sequence of analytic functions  $\{q_j(z)\}_{j \geq 0}$  is called an *inner sequence* if  $\{q_j(z)\}_{j \geq 0}$  consists of inner functions and  $(q_j/q_{j+1})(z)$  is inner for any  $j$ .

We note that the above condition is equivalent to that  $q_j(z)H^2(z)$  is contained in  $q_{j+1}(z)H^2(z)$ . Therefore every inner sequence  $\{q_j(z)\}_{j \geq 0}$  corre-

sponds to a submodule  $\mathcal{M}$  in  $H^2$  as follows:

$$\mathcal{M} = \sum_{j=0}^{\infty} \oplus q_j(z) H^2(z) w^j.$$

In this submodule, we can calculate many subjects of operator theory, exactly.

**Theorem 1 ([2, 3])** *Let  $\mathcal{M}$  be the submodule arising from an inner sequence  $\{q_j(z)\}_{j \geq 0}$ . Then the following hold:*

- (i)  $\| [R_z^*, R_w] \|_2^2 = \sum_{j=0}^{\infty} (1 - |(q_j/q_{j+1})(0)|^2),$
- (ii)  $\| [S_z^*, S_w] \|_2^2 = \sum_{j=0}^{\infty} (1 - |q_{j+1}(0)|^2)(1 - |(q_j/q_{j+1})(0)|^2).$

Let  $q_{\infty}(z)$  be the inner function defined as follows:

$$q_{\infty}(z) H^2(z) = \overline{\bigcup_{j=0}^{\infty} q_j(z) H^2(z)}.$$

Without loss of generality, we may assume that the first non-zero Taylor coefficient of  $q_{\infty}(z)$  is positive.

**Theorem 2 ([2])** *Let  $\mathcal{N}$  be the quotient module arising from an inner sequence  $\{q_j(z)\}_{j \geq 0}$ . Then  $\sigma(S_z) = \sigma(q_0(z))$ , where  $\sigma(q_0(z))$  is the spectrum of  $q_0(z)$ , that is,  $\sigma(q_0(z))$  consists of all zero points of  $q_0(z)$  in  $\mathbb{D}$  and all points  $\zeta$  on the unit circle  $\partial\mathbb{D}$  such that  $q_0(z)$  can not be continued analytically from  $\mathbb{D}$  to  $\zeta$ .*

**Theorem 3 ([2])** *Let  $\mathcal{N}$  be the quotient module arising from an inner sequence  $\{q_j(z)\}_{j \geq 0}$ .*

- (i) *if  $q_m(z) = 1$  for some finite  $m$ , then*

$$\sigma_p(S_w) = \{0\}, \quad \sigma_c(S_w) = \emptyset \quad \text{and} \quad \sigma_r(S_w) = \emptyset,$$

(ii) if  $q_\infty(z) = 1$  and  $q_j(z) \neq 1$  for any  $j$ , then

$$\sigma_p(S_w) = \{0\}, \sigma_c(S_w) = \overline{\mathbb{D}} \setminus \{0\} \text{ and } \sigma_r(S_w) = \emptyset,$$

(iii) if  $q_\infty(z) \neq 1$  and  $q_j(z) \neq q_0(z)$  for some  $j$ , then

$$\sigma_p(S_w) = \{0\}, \sigma_c(S_w) = \partial\mathbb{D} \text{ and } \sigma_r(S_w) = \mathbb{D} \setminus \{0\},$$

(iv) if  $q_j(z) = q_0(z)$  for any  $j$ , then

$$\sigma_p(S_w) = \emptyset, \sigma_c(S_w) = \partial\mathbb{D} \text{ and } \sigma_r(S_w) = \mathbb{D}.$$

Let  $\mathfrak{A}$  denote the weak closed subalgebra generated by  $S_z$ ,  $S_w$  and the identity operator on  $\mathcal{N}$ , and let  $\mathfrak{A}'$  denote the commutant of  $\mathfrak{A}$ .

**Theorem 4 ([2])** *Let  $\mathcal{N}$  be the quotient module arising from an inner sequence  $\{q_j(z)\}_{j \geq 0}$ . Then  $\mathfrak{A} = \mathfrak{A}'$ . Moreover, for any element  $A$  in  $\mathfrak{A}'$ , there exists a sequence of bounded analytic functions  $\{\varphi_j(z)\}_{j \geq 0}$  in  $H^\infty(z)$  such that  $A = \sum_{j \geq 0} S_{\varphi_j(z)} S_w^j$  in the weak operator topology.*

## References

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